Supporting Probabilistic Thinking: A Program for Teachers of 11- and 12-Year-Olds

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People use probabilities to make choices large and small. For example, the woman who grabs an umbrella after hearing there is a 60% chance of rain uses probability, as does the man scrutinizing the supermarket checkout lines to divine which will be fastest. Most people perceive these interpretations as automatic and instinctual. However, probabilistic thinking represents a cognitive feat, combining probability, logic, and problem solving (e.g., MacGillivray, 2018).

Probabilistic thinking—the use of probability and problem-solving to interpret real-life data—develops over childhood (see Bryant & Nunes, 2012). Probability abilities build over time, reaching a critical mass around ages 11–12, when children can consider the likelihood of events and make sense of these likelihoods logically (Piaget & Inhelder, 1951/1975). With support, these children can begin to capitalize on their emerging skills in probabilistic thinking.

At the same time, probabilistic thinking is rarely taught in school explicitly. Probability is typically introduced formally in middle school; for example, probability enters the Massachusetts mathematics curriculum in sixth grade (Massachusetts Department of Elementary and Secondary Education, 2017). These probability topics are often limited to the sample calculation of simple statistics (e.g., calculating an average): they do not include critical thinking or problem-solving tools that incorporate probability, such as evaluating a sample space or comparing probabilities (e.g., Jones, Langrall, Thornton, & Mogill, 1997). In addition, most students do not receive instruction from a mathematics specialist until middle school, and generalist elementary school teachers often need additional tools to help support students' probabilistic thinking skills.

This creates a gap: children acquire abilities in probabilistic thinking by ages 11–12, but they are not taught how to capitalize on these abilities for a year or more, if at all. Students need support in developing their burgeoning probabilistic thinking abilities, and teachers need resources to help guide students' development. A new program seeks to fill this gap.

Program Overview

Supporting Probabilistic Thinking is a new five-unit program designed for teachers of 11- and 12-year-olds—typically fifth and sixth grade—to support the development of probabilistic thinking in their students. The program offers students first-hand experience in learning about six key skills associated with probabilistic thinking. The program also provides explicit instruction in problem-solving strategies that can be used to reason about data in real-world scenarios. By combining probabilistic thinking skills with robust problem-solving strategies, students will be well equipped to practice their developing abilities of probabilistic thinking in the real world.

Essential Skills for Probabilistic Thinking

To reason effectively about probability, one must hone a set of complex, interconnected skills. Many of these skills were first identified by Piaget and Inhelder (1951/1975) in their groundbreaking investigations into children's conceptualization of chance. To assess the development of probability skills in children, Jones and colleagues (1997) synthesized the literature on probability development to create a two-dimensional framework for probabilistic thinking. Jones and colleagues identified four skills associated with probabilistic thinking, and they described several levels of functional understanding for each of these skills. Bryant and Nunes (2012) suggested two additional skills for a total of six skills associated with probabilistic thinking. Accordingly, the program focuses on the six key skills described here.

Understanding the nature of randomness. Randomness lies at the heart of probability. To understand randomness, one must understand that some events occur without a set pattern and without being able to predict one outcome from previous outcomes. For example, a ball dropped from a sufficient height will bounce in a random direction that cannot be predicted. A working understanding of the nature of randomness is essential for probabilistic thinking (Bryant & Nunes, 2012). Moreover, this understanding can help guard against common misconceptions of random phenomena (Batanero, 2015), such as the mistaken belief one can predict future events in a random experiment by observing or counting previous outcomes (i.e., the gambler's fallacy; see Rabinowitz, Dunlap, Grant, & Campione, 1989).

Understanding the sample space. Jones and colleagues (1997) extended the work of Falk (1983) to note that determining the probability of an event is predicated on knowing all the possible outcomes of an experiment. For example, if a fair six-sided die is thrown, one must know the outcome will be a number between one and six, and that each number is equally likely. Understanding the sample space is critical to calculating the probability of an event accurately.

Calculating the probability of an event. Calculating the probability of an event should be simple once the sample space is known: the probability is the number of desired events (the numerator) divided by the number of possible events (the denominator). But this calculation is often fraught, as children often use intuitive or subjective techniques (e.g., Falk, Yudilevich-Assouline, & Elstein, 2012) which lead to inconsistent results (see Jones et al., 1997). Practice in calculating single-event probabilities is needed to promote effective evaluation of uncertainty.

Comparing the probabilities of events. One must frequently choose the most favorable of two uncertain options, such as choosing which of two traffic-riddled streets to use to arrive at a destination most quickly. To do so, one must calculate the probability of each option and then choose the best. Probability depends on a proportional relation, and comparing the probabilities of different events requires comparing proportions (e.g., ratios, fractions; Howe, Nunes, & Bryant, 2011). Learning about such proportions, including fractions, is a hallmark of mathematics education around ages 10–11; students at this age are increasingly well prepared to compare probabilities expressed as ratios and fractions (Howe, Nunes, & Bryant, 2011).

Understanding the correlations between events. Understanding the correlations among various phenomena is important to knowing how the world around us works. Children can understand intuitively that two qualities can be correlated (e.g., children's ages and heights) based on their experience. However, numeric treatments of correlation may prove troublesome; without proper training, children often use naïve problem-solving heuristics to find correlations when data are presented in cross-tabular format (Obersteiner, Bernhard, & Reiss, 2015). Training in evaluating correlations can help scaffold the final skill, using conditional probabilities.

Using conditional probabilities. Understanding conditional probabilities, also known as Bayesian reasoning, represents the pinnacle of probabilistic thinking. Using conditional probabilities, one recognizes when the probability of an event is affected by the occurrence of a related event. Although these intuitive understandings of conditional probability are innate, applying these understandings to formal probability tasks creates problems for learners young and old, leading them to make predictable errors (e.g., Tversky & Kahneman, 1983). When the representation of the task is simplified, however, performance on conditional probability tasks reaches adult-like levels around ages 11–12 (Zhu & Gigerenzer, 2006), again suggesting this age is ripe for training in probabilistic thinking.

Levels of Understanding

All students have some intuitions about these six skills. However, depending on their prior experiences, these intuitions may be fragmented or incomplete. A goal of the program is to increase students' level of understanding of these topics. To assess probabilistic thinking, Jones and colleagues (1997) created a framework with four levels of understanding for each skill. As students ascend through the levels, they show increased knowledge and understandings associated with a skill. The levels are summarized here.

1) *Subjective.* Students apply incomplete understandings of the skill, or they apply intuitive strategies to solve related problems.
2) *Transitional.* Students demonstrate some understanding of the skill, and they use sometimes-effective strategies to solve related problems.
3) *Informal quantitative.* Students have essential understanding of the skill, and they begin to apply effective strategies to solve related problems.
4) *Numerical.* Students have mastery of the skill, and they apply effective strategies to solve related problems with precise numerical solutions.

The present program introduces probability topics and focuses on problem-solving using probabilistic thinking; precise numerical solutions are not emphasized. Accordingly, the program combines Jones and colleagues' (1997) final two skill levels into one: *quantitative.* By using the program, teachers will help students gain mastery in the six key skills of probabilistic thinking and ascend through these levels of understanding.

Mathematical Problem-Solving Strategies

Building mastery in these six skills is necessary for probabilistic thinking to flourish, but not sufficient. To apply the skills effectively, one must employ mathematical problem-solving strategies that link probabilistic thinking skills to real-world scenarios. Polya (1945) offered a robust four-step strategy for addressing any type of mathematical problem. This general structure for mathematical problem-solving lets students apply their knowledge in an orderly, verifiable manner. The four steps in this general problem-solving strategy are shown here.

1) *Understand the problem* by determining exactly what needs to be solved.

2) *Develop a plan* by breaking it into smaller problems or relating it to familiar problems.

3) *Carry out the plan* and check each step while solving.

4) *Check the result* for reasonableness and determine if there are other ways to solve.

Once learned, students may use this general problem-solving strategy is organize their work as they solve increasingly more complex problems featuring probability. Kuhn (2000) argued that conscious awareness of the problem-solving strategies one has available for a given task—*metastrategic awareness*—is an essential component of higher-order cognitive development. Moreover, such metastrategic awareness allows for the ongoing evaluation of which problem-solving techniques are more successful than others, allowing an individual to retire the less effective strategies and adopt better ones (Kuhn, 2005). Metastrategic awareness helps learners think more effectively in all disciplines, including probabilistic thinking. Accordingly, the program encourages students to think explicitly about the problem-solving strategy, as well as the techniques available to solve any probability-related problem.

Understanding Goals

Each of the five units in the program is associated with several understanding goals (Blythe, 1997). In turn, each activity has its own set of sub-goals. These goals are shown here.

Unit 1: Understanding Randomness

The first unit introduces students to probabilistic thinking and to randomness.

Understanding Goals for Unit 1

Students will understand that random events are unpredictable in the short-term but yield predictable long-term patterns; will understand that one cannot predict a purely random event from previous observations; and will begin to use a general mathematical problem-solving strategy to solve complex problems featuring probability.

Activity Sub-Goals for Unit 1

Students will learn about two facets of randomness, short-term unpredictability and predictable long-term patterns; will consider “fairness;” will reinforce their intuitive sense of randomness; will learn that prior observations cannot be used to predict events in a random process; will revisit the probability topics and problem-solving techniques they have learned during the unit; and will engage with the information from multiple perspectives.

Unit 2: Exploring the Sample Space

The second unit invites students to consider all the possible outcomes of an experiment.

Understanding Goals for Unit 2

Students will understand how the sample space is different for different experiments, and how the events within a sample space may not all have the same likelihood of taking place; will understand how to define the sample space for one-stage experiment through listing; and will understand how to define the sample space for two-stage experiments systematically.

Activity Sub-Goals for Unit 2

Students will learn a simple strategy of how to determine if an event is possible, if the sample space is known; will begin to use their imagination to envision a sample space; will learn a technique for enumerating the sample space; will learn to check whether they have listed all the possible events; will learn an extendable technique to find the sample space for multi-stage experiments; will use multiplication to find the total outcomes of a multi-stage experiment; will revisit the probability topics and problem-solving techniques they have learned during the unit; and will engage with the information from multiple perspectives.

Unit 3: Single-Event Probabilities

The third unit gives students practice in calculating the probability of single events.

Understanding Goals for Unit 3

Students will learn how to calculate the probability for an event and express it in a variety of ways; will learn how to determine the most and least likely events, given a scenario; and will learn how to communicate the likelihood of an event.

Activity Sub-Goals for Unit 3

Students will bolster their intuitive understandings of which events are likely and unlikely in several experiments; will use these intuitions to make simple decisions; will calculate probabilities for single-stage experiments; will express these probabilities using numerical expressions; will calculate probabilities for single-stage experiments; will express probabilities using multiple representations; will revisit the probability topics and problem-solving techniques they have learned during the unit; and will engage with the information from multiple perspectives.

Unit 4: Comparing Probabilities

The fourth unit lets students compare different events' probabilities to inform decisions.

Understanding Goals for Unit 4

Students learn how to calculate and compare probabilities for two events; will learn how to avoid common errors in comparing probabilities; and will learn how to use probabilities to inform decision-making.

Activity Sub-Goals for Unit 4

Students will compare probabilities associated with events in a single-stage experiment; will make decisions based on these comparisons; will compare probabilities associated with events in two single-stage experiments with differently sized sample spaces; will avoid common pitfalls when making such comparisons; will observe and summarize data from an experiment;

will compare probabilities to determine whether an experiment is fair; and will engage with the information from multiple perspectives.

Unit 5: Correlation and Conditional Probability

The fifth unit asks students to consider the correlations between phenomena and use this knowledge to calculate conditional probabilities.

Understanding Goals for Unit 5

Students learn how to determine whether two events are correlated; will learn how knowledge about the probability one event can improve knowledge about the probability of a correlated event; and will learn how to use the probability of one event to calculate the conditional probability of another event.

Activity Sub-Goals for Unit 5

Students will make observations to identify correlation; will describe the correlation between events; will describe how one event's probability can be affected by that of another; will calculate conditional probabilities; will use conditional probabilities to make decisions; will see how events can become impossible in non-replacement experiments; and will engage with the information from multiple perspectives.

Learners and Their Learning Challenges

Students ages 11–12 are targeted for the program because their natural probabilistic abilities will be mature enough to respond to instruction. These students are also likely to have received instruction in fractions and ratios, helpful experience when calculating probabilities.

Students may face several challenges in learning probabilistic thinking. First, probability is a highly abstract concept which cannot be observed directly, and students may struggle to find a direct connection between probability and their everyday lives. To combat this, the activities in the app and the classroom use concrete, tangible examples (e.g., choosing outfits, taking photographs). These scenarios are designed to help students see the connections between randomness, probability, and their own experiences.

Second, learning probabilistic thinking—like most skills—takes a great deal of practice to build mastery. If learners were to be given a series of staid mathematics drills, students may lose motivation to proceed. To encourage students to progress through the program, the app provides positive reinforcement as students demonstrate their skills: students earn virtual trophies as they complete activities in the app. This feature is described in the next section.

Teachers may face their own challenges in using the program. Notably, educators without extensive experience in teaching probability may consider the topic only a minor steppingstone to statistical topics which arise in the middle school curriculum (e.g., averages, variability). Although there is little research into the probability knowledge of elementary school teachers (Jones, Langrall, & Mooney, 2007; cited in Chernoff & Russell, 2013), many may not readily see the importance of probability as a mathematical topic or the power of probabilistic thinking in solving real-world problems. The program supports teachers by using concrete, relatable scenarios to teach probabilistic thinking, and by helping students and teachers alike see the importance of probabilistic thinking.

Structure of the Program

The program comprises two components. First, a student-facing tablet application ("the app") lets students explore the six skills of probabilistic thinking and associated problem-solving strategies. The app engages students using real-world scenarios; and using a metacognitive perspective, it trains students to use probabilistic thinking and problem-solving techniques explicitly. Second, a teacher-facing manual ("the manual") gives educators, including those without mathematics specialization, classroom exercises and support for students.

Flipped Classroom Paradigm

The program uses a flipped classroom paradigm: students engage with learning materials independently before the associated topics are presented in the classroom (Bergman & Sams, 2012). Such an arrangement has two distinct advantages in this context. First, the flipped classroom model allows students to explore probabilistic thinking on their own, at their own pace, before engaging in classroom activities: such an arrangement may promote self-efficacy among mathematics students by allowing them to take charge of their learning (Lai & Hwang, 2016). Second, within any size classroom, students' natural abilities of probabilistic thinking and experience with probability in the classroom are likely to be diverse; the flipped classroom allows teachers to focus their attention and feedback in a more personalized manner.

Accordingly, the program is designed with a flipped classroom paradigm. For each of the five units, students first use the app to receive an introduction to a probability topic (e.g., understanding randomness). The app takes students through 3–4 activities on each topic. Then, teachers take students through 1–2 focused classroom activities which build on the same topic.

Student-Facing Application

To create an engaging and effective experience for students, the app uses four design guidelines suggested by learning science researchers for educational software: the app promotes "active, engaged, meaningful, and socially interactive learning" (Hirsh-Pasek et al., 2015, p. 3).

Active learning. Students take an active role in each activity in the app. Activities are framed as "mini-games" within the app: each activity has a game-based objective which must be accomplished through probabilistic thinking. To engage students in each activity, students must students manipulate the tablet to perform certain tasks (e.g., shake the tablet to roll virtual dice, hear auditory feedback when solving problems), not simply view animations or swipe the screen.

The app will also keep learners' minds active by asking them to form mental models of the topics presented, and then to use these models to solve problems. These models accumulate into an armory of problem-solving techniques which students use in later activities—both in the app and the classroom—to solve problems, and students are encouraged to consider these problem-solving techniques explicitly (Kuhn, 2000).

Engagement. The app keeps students engaged by providing frequent, real-time feedback as they solve problems and learn skills. This feedback is designed to promote a growth mindset (e.g., Dweck, 2006), one that celebrates the fact that people can develop their talents through work and perseverance. Accordingly, the feedback is based on the student's efforts, not on the student's perceived intelligence.

Each game-based activity within the app presents its own objective (e.g., escaping from a cave, racing a virtual peer in a car). To advance within the activity, the student must use probabilistic thinking skills (e.g., calculate probabilities, evaluate correlations). When students meet the objectives for an activity, they are awarded a virtual trophy and praised.

To encourage students to engage with the app, the game-based activities are presented along a pre-determined path (a "quest"). When a student completes an activity, the next activity is unlocked. The skills covered by each activity build on previously covered skills. As students complete more activities, they earn additional, virtual trophies, and more game-based apps become available.

Meaningful learning. To be meaningful, a learning app should go beyond basic drills and connect authentically with the learner's experience. In the context of probabilistic thinking, meaningful learning incorporates real-world contexts and active problem solving.

In the real world, problems are complex and often have several different ways to represent and solve them. For example, correlations between two variables can be represented as both cross-tabulations and as scatterplots, and students can learn to use these representations to consider different aspects of the correlations. To provide a rich, meaningful environment for learning probabilistic thinking and problem solving, multiple representations are featured. For example, when considering a single event within a random experiment, the student may be asked to report whether the event is possible or impossible; less likely or more likely; or to report a precise probability using a fraction. The app gives students experience in thinking about these different aspects of probability, and it helps students see the interconnections between them.

Social interaction. The app is designed to be used independently by students: social interaction between students is not a feature of the app. However, the app features feedback from virtual mentors and co-learners, and students are asked to consider this feedback as they progress through activities. For example, when solving a task dealing with the correlation between two events, a virtual mentor may prompt the student to indicate what he or she sees, how the problem might be solved, and what additional information would be helpful to solve it. Even without face-to-face human interaction, students get the experience of sharing their problem-solving strategies with others, helping make these strategies explicit from a metacognitive perspective.

Teacher-Facing Manual

The manual, like the app, is structured around the six key skills of probabilistic thinking. In five units, the manual provides interactive classroom activities and pedagogical notes associated with the key skills and their associated problem-solving strategies. For each unit, the manual provides one or two classroom activities that reinforce topics introduced by the app (e.g., sample space). These activities reinforce students' metacognitive awareness of problem-solving strategies, and they use social interaction to help students learn to communicate their skills.

Metacognitive problem-solving strategy. To help students organize their work in complex, multi-step tasks, each classroom activity calls on students to use the general four-step problem solving strategy (see Polya, 1945). By modeling thoughtful problem-solving strategies, teachers can help reinforce the importance of considering the techniques available to students when solving problems.

Social interaction. Each classroom activity asks students to work in groups to solve complex problems. By asking students to collaboratively to explore and to solve multi-step problems, students gain experience in communicating their probabilistic thinking skills.

Assessment Plan

The app and the manual offer several ways for students to assess their own learning and for teachers to assess their students' learning. First, the app displays students with feedback on their progress through the game-based activities. Specifically, the app awards students with trophies as they attempt activities, and additional activities of increasing complexity are revealed on the "quest" screen as students complete activities. This feedback lets students see how much progress they have made through the program, and it helps motivate them to continue learning.

Second, the app asks students to reflect on their own learning. At the end of each unit, the app presents students with a "reflection activity." This activity asks students to (a) solve multi-step probability problems using the general problem-solving strategy, and (b) to reflect on what they have learned during the unit. By completing this activity, students can consider the growth they have made during the unit, and they can consider their areas for continued growth.

Third, the program helps teachers assess their students' learning. To gauge students' motivation and engagement with the program, teachers can inspect students' progress using the "quest" screen, just as students can. In addition, teachers can observe students' participation in the classroom activities which conclude each unit: each classroom activity lets students demonstrate their understanding of the material through a variety of performances (e.g., writing the solutions to complex problems). Lastly, teachers can read students' responses from the "reflection activities" to gauge the level of understanding students have of each skill (as described previously in this document). By reading students reflection responses, and by comparing these responses with the probabilistic thinking framework, teachers can assess how students' understandings have grown, and this information can be used to tailor learning experiences in the classroom.

Discussion

Supporting Probabilistic Thinking will help 11- and 12-year-olds develop robust probabilistic thinking skills, both on their own and with the help of their teachers. However, as with any new classroom intervention, the program has several strengths and challenges.

The program uses a flipped classroom paradigm to promote student self-efficacy and to allow teachers to individualize their attention and feedback in the classroom. However, the relative newness of this paradigm may be unfamiliar to some educators, requiring a shift within the minds of educators (e.g., de Araujo, Otten, & Birisci, 2017). Indeed, learning outcomes in flipped classrooms may not always be superior to those using traditional non-flipped formats (e.g., DeSantis, van Curen, Putsch, & Metzger, 2015). Additional research would be needed to determine whether the flipped classroom paradigm is superior with highly abstract topics like probabilistic thinking.

The program purposefully focuses on probability, a branch of mathematics, and not on the related science of statistics. Although this focus is tacit, it is pervasive in the program:

students are not asked to complete statistics tasks (e.g., calculating averages) to keep the focus on probability. However, some statistical skills are complementary to those covered by the program: for example, critical to statistics is the ability to record experimental observations and summarize data. Because of the limited scope of the program, these complementary skills are not addressed. However, future iterations of the program may include optional extensions to address them.

Conclusion

As children rise from elementary school to middle school, their brains are ready to engage with probability in new and exciting ways. These children now begin to understand how probability shapes the world around us, and they can begin to reason with this newfound information. By employing robust problem-solving skills, they can apply their probabilistic thinking skills to new scenarios and use data to inform predictions about real-world events.

Dedicated practice is required to develop and hone probabilistic thinking skills. This program will help 11- to 12-year-olds explore probability by providing a rich app-based environment in which to practice their nascent probabilistic thinking skills. The program will also offer elementary school teachers a vibrant set of classroom activities and resources, allowing teachers to use the program to reinforce probabilistic thinking and problem-solving concepts. When used together, the app and the manual will help children see how probability shapes our world, empowering students as they learn how to reason about real-world data effectively.

References

Batanero, C. (2015, February). Understanding randomness: Challenges for research and teaching. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the Ninth Congress of European Research in Mathematics Education* (pp. 34–49). https://hal.archives-ouvertes.fr/hal-01280506

Bergman, J., & Sams, A. (2012). Flip your classroom: Reach every student in every class every day. International Society for Technology in Education.

Blythe, T. (1997). *Teaching for understanding guide.* Jossey-Bass.

Bryant, P., & Nunes, T. (2012). *Children's understanding of probability: A literature review (full review)*. Nuffield Foundation.

Chernoff, E. J., & Russell, G. L. (2013). Comparing the relative likelihood of events: The fallacy of composition. In M. Martinez & A. Castro Superfine (Eds.), *Proceedings of the 35th annual meeting of the North American chapter of the international group for the psychology of mathematics education.* University of Illinois at Chicago.

de Araujo, Z., Otten, S., & Birisci, S. (2017). Mathematics teachers' motivations for, conceptions of, and experiences with flipped instruction. *Teaching and Teacher Education, 62*, 60–70. https://doi.org/10.1016/j.tate.2016.11.006

DeSantis, J., van Curen, R., Putsch, J., & Metzger, J. (2015). Do students learn more from a flip? An exploration of the efficacy of flipped and traditional lessons. *Journal of Interactive Learning Research, 26*(1), 39–63. https://www.learntechlib.org/primary/p/130133/

Dweck, C. S. (2006). *Mindset: The new psychology of success*. Random House.

Falk, R. (1983). Children's choice behaviour in probabilistic situations. In D. R. Grey, P. Holmes, V. Barnett, & G. M. Constable (Eds.), *Proceedings of the First International*

Conference on Teaching Statistics, Volume II (pp. 714–716). International Association of Statistical Education.

Falk, R., Yudilevich-Assouline, P., & Elstein, A. (2012). Children's concept of probability as inferred from their binary choices—revisited. *Educational Studies in Mathematics, 81*(2), 207–233. https://doi.org/10.1007/s10649-012-9402-1

Hirsh-Pasek, K., Zosh, J. M., Golinkoff, R. M., Gray, J. H., Robb, M. B., & Kaufman, J. (2015). Putting education in "educational" apps. *Psychological Science in the Public Interest, 16*(1), 3–34. https://doi.org/10.1177/1529100615569721

Howe, C., Nunes, T., & Bryant, P. (2011). Rational number and proportional reasoning: Using intensive quantities to promote achievement in mathematics and science. *International Journal of Science and Mathematics Education, 9*(2), 391–417. https://doi.org/10.1007/s10763-010-9249-9

Jones, G. A., Langrall, C. W., Thornton, C. A., & Mogill, A. T. (1997). A framework for assessing and nurturing young children's thinking in probability. *Educational Studies in Mathematics, 32*(2), 101–125. https://doi.org/10.1023/A:1002981520728

Kuhn, D. (2000). Metacognitive development. *Current Directions in Psychological Science, 9*(5), 178–181. https://doi.org/10.1111/1467-8721.00088

Kuhn, D. (2005). *Education for thinking*. Harvard University Press.

Lai, C. L., & Hwang, G. J. (2016). A self-regulated flipped classroom approach to improving students' learning performance in a mathematics course. *Computers and Education, 100*, 126–140. https://doi.org/10.1016/j.compedu.2016.05.006

MacGillivray, H. (2018). Real probability and probabilistic thinking. *Teaching Statistics, 40*(2), 37–39. https://doi.org/10.1111/test.12159

Massachusetts Department of Elementary and Secondary Education. (2017). *Massachusetts curriculum framework – 2017: Mathematics.* http://www.doe.mass.edu/frameworks/math/2017-06.pdf

Obersteiner, A., Bernhard, M., & Reiss, K. (2015). Primary school children's strategies in solving contingency table problems: The role of intuition and inhibition. *ZDM, 47*(5), 825–836. https://doi.org/10.1007/s11858-015-0681-8

Piaget, J., & Inhelder, B. (1975). *The origin of the idea of chance in children* (L. Leake, P. Burrell, & H. D. Fishbein, Trans.). Norton. (Original work published 1951)

Polya, G. (1945). *How to solve it: A new aspect of mathematical method.* Princeton University Press.

Rabinowitz, F. M., Dunlap, W. P., Grant, M. J., & Campione, J. C. (1989). The rules used by children and adults in attempting to generate random numbers. *Journal of Mathematical Psychology, 33*(3), 227–287. https://doi.org/10.1016/0022-2496(89)90009-6

Tversky, A., & Kahneman, D. (1983). Extensional versus intuitive reasoning: The conjunction fallacy in probability judgment. *Psychological Review, 90*(4), 293–315. https://doi.org/10.1037/0033-295X.90.4.293

Zhu, L., & Gigerenzer, G. (2006). Children can solve Bayesian problems: The role of representation in mental computation. *Cognition, 98*(3), 287–308. https://doi.org/10.1016/j.cognition.2004.12.003