Supporting Probabilistic Thinking

A Program *for* Teachers of 11- and 12-Year-Olds

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Introduction

Every day, people interpret probability to make choices. The woman who grabs an umbrella after hearing there is a 60% chance of rain uses probabilities, as does the man figuring which checkout line at the supermarket might be fastest. People perceive these thoughts as automatic. But ***probabilistic thinking*** is an incredible mental feat, synthesizing knowledge of probability, problem-solving, and evidence.

Decades of research have revealed that our abilities to think about probability mature over childhood. These abilities reach adult-like levels around ages 11–12. In 5th and 6th grade, children are ready to learn about probability and uncertainty. However, most students do not receive instruction in probability until 6th or 7th grade.

This creates an opportunity. 11- and 12-year-olds will spend a lifetime reasoning with probability, and their young minds are ready to learn about probability. By learning to use probability as an everyday thinking tool, these children will be better equipped to deal with life's uncertain events.

About this program

This program, SUPPORTING PROBABILISTIC THINKING, has two main components:

1) ***This resource for teachers.*** This manual presents a framework for probabilistic thinking and details a five-unit program to support 11- and 12-year-olds as they develop probabilistic thinking skills.

2) ***The tablet-based application for students.*** The SPT App guides students through the same five-unit program. The app introduces each topic and lets students explore them at their own pace before the classroom activities. Appendixes A and B provide more information about the app.

The program provides a gentle introduction to probabilistic thinking. Intuitive approaches are embraced whenever possible: no special mathematical or superhuman abilities are assumed.

Only two assumptions are made: probability is an essential component of thinking about the world, and everyone—*everyone*—can learn to think about it effectively.

Plenty of support is provided along the way, both here and at **probabilisticthinking.com**.

What is probabilistic thinking?

Probabilistic thinking can be considered in two ways: **intuitively** and **quantitatively**.

Intuitively, children start thinking using probability from a very early age. Influential child psychologists Jean Piaget and Bärbel Inhelder first wrote about probabilistic thinking in children as "a sort of spontaneous estimate of the more or less probable character of feared or expected events" (1951/1975, p. xv). From this perspective, probabilistic thinking is a gut-level way of thinking about events which may happen but are not certain.

In contrast, most academic treatments of probability take a markedly quantitative perspective. Students are trained to use fractions and ratios to quantify the likelihood of uncertain events, often without a real understanding of probability.

Clearly, students must have both intuitive and quantitative relationships with probability. That is, students must have an intuitive understanding of chance, and also have a way of quantifying probability for use in real-world situations.

To balance these perspectives, probabilistic thinking can be defined in pragmatic terms, such as the ability to consider "real data and contexts, understanding conditional probability through both data and chance … [and] understanding strength of evidence, risk, assumptions and independence" (MacGillivray, 2018, p. 37). This definition provides for both intuitive understandings of chance as well as reasoned, data-based calculations of probability.

This program uses a balanced approach to consider probabilistic thinking. As such, SUPPORTING PROBABILISTIC THINKING defines probabilistic thinking in this way:

Students use **probabilistic thinking** when they use intuitive and quantitative ideas of chance and apply them to real-world scenarios using logic and mathematical problem-solving techniques.

Key learning goals for students

SUPPORTING PROBABILISTIC THINKING has four key learning goals for students:

- Students will build on their intuitive understandings of probability and use them to make effective decisions.
- Students will be able to apply basic techniques of quantifying probability to help them solve problems.
- Students will join their knowledge of probability with sound mathematical problem-solving techniques.
- Students will use probabilistic thinking as a thinking tool when evaluating real-world contexts involving uncertainty.

Framework for probabilistic thinking

Effective probabilistic thinking requires knowledge of several interlinked topics. As shown in Figure 1, there are six topics associated with probabilistic thinking; these topics are reflected in the five units of this program. As students gain skill in each topic, they move across three skill levels, represented in the framework as columns.

General problem-solving strategy

Probabilistic thinking joins probability topics with mathematical problem-solving. Figure 2 presents a robust problem-solving strategy that can be applied to a variety of mathematics problems, including those presented here. More information about the strategy is shown in Unit 1.

Figure 1. Framework for probabilistic thinking

Topic	Level 1: Subjective	Level 2: Transitional	Level 3: Quantitative
Randomness	Sometimes recognizes random events. Uses subjective reasoning to identify randomness.	Sometimes identifies experiments which yield random results.	Uses valid reasoning to identify and describe experiments which yield random results.
Sample Space	Lists some outcomes for a one-stage experiment.	Lists all outcomes for a one-step experiment and some for a two-stage experiment.	Uses a generative strategy to list outcomes of a two-stage experiment.
Single-Event Probabilities	Predicts most/least likely events using subjective criteria. Sometimes identifies impossible events.	Predicts most/least likely events using objective criteria. Identifies impossible and possible events.	Predicts most/least likely events using numeric criteria. Calculates numerical probability of an event.
Comparing Probabilities	Compares probabilities using inconsistent or subjective reasoning.	Compares probabilities in and begins to use valid reasoning in simple scenarios. Sometimes distinguishes fair vs. unfair probability experiments.	Compares probabilities numerically and uses valid reasoning to justify choices. Distinguishes fair vs. unfair probability experiments.
Correlation	Identifies some events which tend to occur together.	Sometimes uses valid reasoning to describe and explain simple correlations.	Describes correlations by sign (e.g., positive) and uses valid reasoning to explain them.
Conditional Probability	Following a particular outcome, overgeneralizes to predict that it will definitely occur next time or that it will never reoccur.	Sometimes recognizes how probabilities change, including the creation of impossible events, in non-replacement experiments. Identifies some independent events.	Recognizes how probabilities change in non-replacement experiments. Calculates conditional probabilities in simple situations. Identifies independent events.

Adapted from Jones, Langrall, Thornton, and Mogill (1997), incorporating work by Bryant and Nunes (2012)

Figure 2. Four-step mathematical problem-solving strategy

#	Step	Description
First	Understand the problem	Figure out what is known and what is needed. Use pictures and words.
Second	Develop a plan	Find a similar problem or break it into simpler problems.
Third	Carry out the plan	Carry out the problem and check each step along the way.
Fourth	Check the result	Check the solution for reasonableness. Check for other ways to solve.

Adapted from Polya (1945)

Unit 1: Randomness & Problem Solving

Randomness undergirds uncertainty. It is all around us, but it can be elusive and hard to define. Unit 1 introduces students to the SUPPORTING PROBABILISTIC THINKING program, allows students to explore several facets of randomness, and empowers students with a robust problem-solving strategy.

Students can start using the SPT App right away. The app allows students to explore independently. Using the flipped classroom model, students encounter concepts in the app which are amplified and reinforced through subsequent classroom activities.

Learning goals for Unit 1

By the end of Unit 1, students will learn:

- random events are unpredictable in the short-term but yield predictable patterns in the long-term
- one cannot predict a purely random event from previous observations
- a general mathematical problem-solving strategy to solve complex problems

In the student app

Students are encouraged to use the SPT App before the unit is addressed in class. The first two activities introduce students to the SPT App and let students explore the concept of randomness. The third activity, *Reflection*, is repeated at the end of every unit: it allows students to reflect on the concepts they have learned during the unit.

The Unit 1 app activities are summarized here. Detailed descriptions are presented in Appendix B: Student App Activities.

App Activity 1.1: Introduction

Students are introduced to the SPT App, to probabilistic thinking, and to randomness.

App Activity 1.2: Shipments

Two facets of randomness are explored: short-term unpredictability and predictable patterns over the long-term.

App Activity 1.3: Laundry

Two facets of randomness are reviewed, and the gambler's fallacy is introduced.

App Activity 1.4: Reflection

Students reflect on what they have learned in the SPT App in their own words.

In the classroom

Two interrelated activities introduce students to randomness and a general problem-solving strategy. The activities also give students a preview of topics to be addressed in future units (e.g., single-event probability).

Classroom Activity 1.5: Fair Game

In this activity, students work in pairs to explore randomness and its connection to the concept of “fairness.” This activity is adapted from *Fair Games, Unfair Games* by Bright, Harvey, and Wheeler (1981).

- Divide students into pairs. Each pair will need a pair of dice and scratch paper.

- Tell students that they will play two games. Their job is to figure out which game is fairer. Ask students for their definition of a fair game. (Listen for: A game where each person has an equal chance to win.)

- Each pair should have one person be on Team Even, and the other on Team Odd.

- **Game 1: Sums.** In the first game, the pair takes turns rolling the dice. After each roll, the pair should add the two numbers together. If the sum is even, Team Even earns a point; if odd, the point goes to Team Odd. Instruct students to roll to use their scrap paper to keep score. The winner is the Team with the most points.

- Let the pairs play. Call time after two minutes and ask the pairs to find the winner.

- **Game 2: Products.** Again, the pair should take turns rolling. This time, the pair should find the product of the two numbers by multiplying them.

- Let the pairs play. After two minutes and ask the pairs to find the winner.

- Ask each pair which game they think is fairer and why. Encourage students to review their tally sheets ask they answer.

- Introduce the general problem-solving strategy to students:

    - Understand the problem
    - Develop a plan
    - Carry out the plan
    - Check the result

- **Understand the problem.** Ask students to reiterate the problem they are trying to solve. (Choosing which game is fairer.) What does it mean for a game to be fair? (Each player has an equal chance to win.)

- **Develop a plan.** What are ways to check the fairness of each game? (See how many times Team Even won vs. Team Odd. Map out the possible outcomes of each game.) How could someone map out the possible outcomes of each game? (Making a table of the possible rolls and results.)

- **Carry out the plan.** Take a poll to see how many times Team Odd won vs. Team Even for each game. Then make a 6x6 table for each game and fill in the sums for Game 1 and the products for Game 2. Ask students which game gives an advantage to Team Even.

- **Check the result.** The tables show that Team Even and Team Odd should both win Game 1 about half the time, but that Team Even will win Game 2 about three-quarters of the time. Check to see that students’ experience was similar.

- Ask students if it is *possible* for Team Odd to win Game 2, and if it is *likely*. (It is possible but unlikely.)

- Ask students if Team Even and Team Odd always have to get the same score in Game 1. (No, they do not.) Ask students to reason why. (Each roll of the dice is unpredictable/random.)

Classroom Activity 1.6: The Chance Bowl

In this activity, students will expand on the idea of "fairness" as they explore randomness.

- Many games and sporting events start with a coin flip to see who goes first. Ask students why they think a coin flip is used. (It is a fair game. Each side will win about half the time.)

- Another common game-starter is the roll of a die, and the higher value goes first.

- Now, students should work in small groups to develop a new way to start a new sporting event, The Chance Bowl. The Chance Bowl has to start with a simple, fair game, but it can't start with a coin flip or a roll of the dice.

- Students should brainstorm a new way to start The Chance Bowl.

- Each team should demonstrate that their technique is fair by using the problem-solving strategy.

    - *Understand*. Describe what makes a game fair.
    - *Plan*. Try the technique to make sure both sides win about the same number of times. Write out the possible outcomes and check that each side should win about half the time.
    - *Carry out*. Execute the plan.
    - *Check*. Check that the results make sense, and that the observed results are similar to the expected results.

- Invite the teams to launch The Chance Bowl in front of the class by using their invented technique. One student should introduce the technique, two students should pretend to be rival teams and execute the technique, and one student should briefly describe why the game is fair.

Unit 2: Defining the Sample Space

When flipping a coin, there are two possible outcomes: heads and tails. Dice have six possible outcomes when rolled. These possibilities are the ***sample space*** of an experiment.

To consider the probability of an event, one must understand the sample space. The sample space is sometimes simple to define; other times, strategy must be used.

Learning goals for Unit 2

In this unit, students will learn:

- how the sample space is different for different experiments, and how the events within a sample space may not all have the same likelihood of taking place
- how to define the sample space for one-stage experiment through listing
- how to define the sample space for two-stage experiments systematically

In the student app

As always, students are encouraged to use the student app to explore the unit before it is discussed in class.

App Activity 2.1: Impossible

Students consider several experiments and decide whether an event is possible (i.e., in the sample space) or impossible.

App Activity 2.2: Listings

For a series of experiments, students list the possible outcomes systematically.

App Activity 2.3: Frankenstein

Students learn a systematic technique to list the possible outcomes of an experiment.

App Activity 2.4: Reflection

Students reflect on what they've learned in the first two units.

In the classroom

Defining the sample space is a critical skill for calculating probability (which is covered in the next unit). This classroom activity helps students consider the sample space quickly, even with multi-stage experiments.

Classroom Activity 2.5: Chance Couture

In this activity, students work together to solve problems about the sample space.

- Students will imagine a new social media influencer who is known for having a huge wardrobe. Ask students to brainstorm a name and favorite charity for the influencer.
- The influencer has five shirts in the closet: red, orange, yellow, green, and white. Remind students that this means there are five possibilities in the sample space: one shirt of each color.

- The influencer has three pairs of pants in the closet: purple, khaki, and black.
- The influencer gets dressed in a strange way: they reach into their closet without looking and grab the first things they touch. Ask a student to act out how the influencer might get dressed.
- The influencer wants to know how many possible outfits can be made from these five shirts and three pairs of pants. Encourage students to solve the problem using the general problem-solving strategy.
- **Understand the problem.** Ask students to reiterate the problem they are trying to solve. (How many shirt-pants combinations are there?)
- **Develop a plan.** What are ways to count the combinations? (List them out using a grid. Multiply the numbers of shirts and pants.)
- **Carry out the plan.** As students to list out the combinations using a grid and to count how many combinations there are.
- **Check the result.** The number of combinations in the grid should equal the product of the number of shirts and pants (5x3=15).
- Remind the students that the influencer gets dressed randomly: they reach into the closet and put on the first shirt and pair of pants they touch. Ask them: are any combinations of clothes (outfits) more likely to happen than others?
    - The correct answer is no, the outfits are all equally likely.
    - Students may say some outfits are less likely because the colors clash. The influencer dresses randomly, so each outfit has an equal likelihood, even ugly ones.
- The influencer gets a new shirt from a fan: a silver shirt. How many possible outfits does the influencer have?
    - Encourage students to use the problem-solving technique.
    - Students should be able to extend the grids they made earlier and to compare the result with multiplication (6x3=18).
- The influencer gets a sponsorship deal with a sneaker company and has two pairs of new shoes: aqua and grey. They need to have a pair of shoes in each outfit now: an outfit is now a shirt, a pair of pants, and a pair of shoes. How many combinations (outfits) are there?
    - Encourage students to use the problem-solving technique.
    - Students can extend the grids they have made, creating one grid to go with the aqua shoes and one with the grey shoes.
    - Students can also use a mapping technique to list out all the combinations.
    - Students should compare their results with multiplication (6x3x2=36).

Unit 3: Single Event Probability

An intuitive understanding of whether something is likely to happen or not is extraordinarily useful. On occasion, it also helps to quantify these likelihoods to make decisions. This unit focuses on intuitive and quantitative aspects of probability for single events.

Unfortunately, most probability instruction begins here: the calculation of single-event probabilities. As shown so far in SUPPORTING PROBABILISTIC THINKING, there are many precursor understandings can help students truly understand what probability means. In this unit, students apply their knowledge from the first two units and they discover the probability of single events.

Learning goals for Unit 3

In this unit, students will learn:

- how to calculate the probability for an event and express it in a variety of ways
- how to determine the most and least likely events, given a scenario
- how to communicate the likelihood of an event

In the student app

As always, students are encouraged to use the student app to explore the unit before it is discussed in class.

App Activity 3.1: Unlikeliest

Students practice intuitions of probability by describing events as likely or unlikely.

App Activity 3.2: Caves

Students calculate probabilities for single-stage experiments and express them as fractions.

App Activity 3.3: Explorer

Students calculate probabilities for straightforward two-stage experiments and express them as fractions.

App Activity 3.4: Reflection

Students reflect on what they've learned in in the SPT program so far.

In the classroom

In the first activity, students practice calculating probabilities in a multi-stage experiment. In the second activity, students use observed data to communicate probabilities in multiple ways.

Classroom Activity 3.5: Last Level, Part 1

In the SPT App, students worked on an adventure game. Now, the class will imagine how the game might end.

- In the game, players can end up on another planet, on top of a mountain, or at the bottom of an ocean. Ask the students to name three planets, three mountains, and two oceans, such as:

Planets (*Mars, Venus, Saturn*)
Mountains (*Everest, K2, Kilimanjaro*)
Oceans (*Pacific, Atlantic*)

- The player can also end up with one of two jobs: farmer or musician. Ask students to name two things that are farmed and three music genres, such as:

  Farmer (*buffalo, mung beans*)
  Musician (*K-pop, metal, smooth jazz*)

- Tell students that in the game, the player's choices lead them to one of the eight planets, mountains, or oceans; and that each is equally likely. Ask the students what "equally likely" probably means here. (Each has an equal chance of being selected.)

- Remind students that at the end of the game, players can wind up in one of the eight environments and with one of the five jobs. How many endings are there?

  - Encourage students to use the problem-solving strategy to solve the problem.

  - **Understand the problem.** Ask students to reiterate the problem. (How many endings or environment-job pairs are there?)

  - **Develop a plan.** Ask the students if this problem is like any they have solved before. (Yes, it is like the sample space problems. It can be solved by making a grid of all the possible pairs.)

  - **Carry out the plan.** Ask students to draw out a grid to solve the problem. Then invite students to share their work on the board.

  - **Check the result.** Ask students to report the number of combinations. It should equal the product of the number of environments and jobs (8x5=40).

- Ask students to illustrate each of the 40 environment-job combinations. Students should illustrate each combination on a separate piece of paper for a total of 40 papers. Students should be sure to label the combination at the top of the page.

- Collect the illustrations and place them into a container.

- Remind students that a player's choices in the game lead them randomly to one of the 40 endings. Is it equally likely that the game will end on top of a mountain as at the bottom of the ocean?

  - Encourage students to use the problem-solving strategy to solve the problem.

  - **Understand the problem.** Ask students to reiterate the problem. (Is it equally likely that the game will end on top of a mountain as at the bottom of the ocean? Is one more likely than the other?)

  - **Develop a plan.** Ask the students if this problem is like any they have solved before. What is the best way to solve it? (Count the number of mountain endings and compare with the number of ocean endings.)

  - **Carry out the plan.** Ask students to count the mountain endings and the ocean endings. Then ask students to come up and use the grid on the board to count.
  - **Check the result.** Ask students to answer the question. It is more likely that the game will end on top of a mountain (15 out of 40 endings) than at the bottom of the ocean (10 out of 40 endings).

- To conclude the activity, ask students to report the probability of endings where the player is…
  - …a farmer (16 out of 40)
  - …on another planet (15 out of 40)
  - …a farmer on another planet (6 out of 40)

Classroom Activity 3.6: Last Level, Part 2

When using probabilistic thinking, we often have to work with actual, real-world data. This activity extends previous one to incorporate empirical data. Then, students are asked to communicate probabilistic information in several ways.

- Remind students that a player's choices in the game lead them randomly to one of 40 endings. Earlier, they found it was more likely to finish atop a mountain than at the bottom of the ocean.
- Get the container with students' illustrations of the 40 endings. Ask them if they can predict which ending you will pick if you choose one of the illustrations from the container without looking. (No, a random selection is not predictable.)
- Ask students what kinds of predictions they could make if the game was played many times (i.e., if you made many random selections).
  - Predictions might include: more mountain endings than ocean endings, fewer farmer endings than musician endings.
- Ask students to come up one at a time and choose an ending randomly from the container. They should mark their choice on the grid (still on the board) and place their selection back in the container.
- Ask students if their predictions were borne out by the data. Why or why not?
- Distribute the endings randomly to students. Ask students to write a letter to the player, to be shown on screen at the end of the game, that includes:
  - The details of the ending (e.g., name of environment, job)
  - How likely the ending was (using "___ out of __" form).
  - The probability of ending up in that environment, and the likelihood of landing that job
  - The details of a more probable ending than the one the player got, and of a less likely ending
  - Congratulations for finishing the game

Unit 4: Comparing Probabilities

Once someone knows how to find the probability of a single event, they can begin to practice a key skill of probabilistic thinking: comparing probabilities to inform decision-making.

Comparing probabilities can be difficult: some common errors in reasoning can get in the way of these comparisons. In this unit, students will learn how to compare probabilities effectively and them to make better decisions.

Learning goals for Unit 4

In this unit, students will learn:

- how to calculate and compare probabilities for two events
- how to avoid common errors in comparing probabilities
- how to use probabilities in decision-making

In the student app

As always, students are encouraged to use the student app to explore the unit before it is discussed in class.

App Activity 4.1: Recycling

Students compare the probability of various events to make decisions.

App Activity 4.2: Fishing

Students compare probabilities of events in experiments of differently sized sample spaces to make decisions.

App Activity 4.3: Malfunction

Students examine data and compare probabilities to determine whether an experiment is fair.

App Activity 4.4: Reflection

Students reflect on what they've learned about probabilistic thinking so far.

In the classroom

Comparing probabilities is an essential component of probabilistic thinking. In these classroom activities, students will revisit the meaning of probability, and they will use probabilities to make decisions.

Classroom Activity 4.5: Good Behavior

In this activity, gather empirical information to estimate probabilities, and then they make decisions using data.

- In another class, a teacher rewards students' good behavior in class using a six-sided die. When a student behaves particularly well, she rolls the die to pick a reward. The possible rewards are:

    A. Extra five minutes of recess
    B. Gold star next to name on board
    C. Pick next book for class to read

- Recess is selected if a 1, 2, or 3 is rolled. A gold star is selected for a 4 or 5, and choosing the book is selected for a 6.

- Ask students to report the probability of the three possible options:

    - Five minutes of recess (3 out of 6)
    - Getting a gold star (2 out of 6)
    - Picking the book (1 out of 6)

- Count the students in class. Ask students to imagine that the teacher rolled the die once for each student. Based on these probabilities, how many students would be *expected* to get a gold star?

    - The probability of getting a gold star is 2 out of 6, which is $^2/_6$ or $^1/_3$.

    - Multiply the number of students in the class by $^1/_3$ to find the answer.

- Ask students whether this means that exactly one-third of the class would definitely get a gold star. (No, it is random, but $^1/_3$ would be expected.)

- Tell students that another class uses the same approach, but the teacher uses a different die. Different numbers are linked to the same rewards. Whenever the teacher gives a student a reward, he rolls the die behind a screen so no one else can see it.

- Students want to know how many faces of the die are associated with recess, how many with the gold star, and how many with the book. What is the best way for them to do it?

    - Encourage students to use the problem-solving strategy to solve the problem.

    - **Understand the problem.** Ask students to reiterate the problem. (How many faces are associated with each reward? What is the probability of getting each reward using the die?)

    - **Develop a plan.** Ask the students if this problem is like any they have solved before. (Yes, it is like the conveyor belt problem in SPT App. They should record data, find patterns, and then decide.)

    - Ask students to discuss how they might collect data. (For many rolls of the dice, record which rewards were selected, then look for patterns.)

    - Ask students to report how many "gold stars" they would expect out of 60 if that were the option linked to one face of the die. (10 out of 60.) On two faces? (20 out of 60.) Ask a student to write these expectations on the board, plus expectations for 3, 4, 5, and 6 faces.

    - **Carry out the plan.** Invite three more students to the board to record data. Behind a screen, roll the dice and announce the reward (see below). Ask students to tally the rewards as they are announced.

    - The following plan is suggested: recess if a 1 is rolled; gold star if a

2, 3, or 4 are rolled; and the book if 5 or 6 are rolled.

  - *Note*: If performing the experiment in class is not convenient, tell students that the other class made the following data table:

    Recess: 8 out of 60
    Gold star: 31 out of 60
    Book: 21 out of 60

  - Ask the students to compare the results with their expectations to determine how many faces of the die are probably associated with "gold star." (3 out of 6)
  - **Check the result.** Ask students to recommend other ways of answering the same question that might be used to check the result. (Examples: make another set of observations, look at the die itself.)

Classroom Activity 4.6: TukTuk

In this activity, students practice their ability to compare probabilities and to communicate probabilities succinctly.

- On TukTuk, the hot new social network, people from all over the world use their phones to record themselves dancing.
- Two of the network's biggest stars are having contests to find new and talented TukTuk users.
- Both contests work in the same way. To enter the contest, an entrant posts a video of them doing the TukTuk star's signature dance. Winners are selected randomly from the first people to post.
  - Chi Chi D'Amigo will pick 10 winners randomly from the first 1,000 posts.
  - Lauren Beige will pick 50 winners randomly from the first 10,000 posts.
- Ask the class to brainstorm the type of dance and music that each TukTuk star uses (e.g., robot dancing to rock and roll, salsa dancing to drum and bass).
- Tell your students that you really want to win one of these two contests, but you only have time to rehearse for one dance and make one post. Which contest do you have the better probability of winning?
- Ask students to write a Tweet to you that includes the following information:
  - Odds of winning each contest (10/1000 or 1/100; 50/10000 or 1/200)
  - Name of the star's account that you should enter and why (Chi Chi D'Amigo, the chance of winning is double)
  - One dance tip or move that will *really* get everyone's attention

Unit 5: Correlation & Conditional Probability

Everyday events are rarely isolated. For example, when rain has made the sidewalk wet, we know slipping is more likely; this might prompt us to change shoes. This decision uses ***conditional probability***, the likelihood that one event will happen (i.e., slipping on the sidewalk) when we already know another event has happened (i.e., rain made it wet).

Conditional probability sits at the pinnacle of probabilistic thinking. This unit lets students consider correlations between events—an essential precondition for conditional probability—and then helps them explore conditional probability from intuitive and quantitative perspectives.

Learning goals for Unit 5

In this unit, students will learn:

- how to determine whether two events are correlated
- how knowledge about the probability one event can improve knowledge about the probability of a correlated event
- how to use the probability of one event to calculate the probability of another

In the student app

As always, students are encouraged to use the student app to explore the unit before it is discussed in class.

App Activity 5.1: Spaceship

Students consider the meaning of correlations between related events.

App Activity 5.2: Envoy

Students explore conditional probability in non-replacement experiments.

App Activity 5.3: Diplomacy

Students calculate conditional probabilities and use them to inform decisions.

App Activity 5.4: Reflection

Students reflect on what they've learned during SUPPORTING PROBABILISTIC THINKING.

In the classroom

Conditional probabilities use information we already have to make better decisions about events yet to come. Doing so draws on all facets of probabilistic thinking.

Classroom Activity 5.5: Bending the Curve

In these final classroom activities, students will use all of their probabilistic thinking skills to explore the fight against COVID-19.

- COVID-19, the disease caused by the novel coronavirus, continues to threaten people around the world. But scientists are working hard to track the disease, develop therapies, and create a vaccine.
- Divide the class into small groups. Each

group will imagine fighting COVID-19 in a U.S. state. Ask each group to select a state and share it with the class.

- As of May 2020, the USA has 330 million people (U.S. Census Bureau, 2020), and 1.3 million people have tested positive for coronavirus (New York Times, 2020).
- When scientists study diseases, they often measure how many people out of 100,000 have the disease. Ask the groups how they would figure out how many people per 100,000 in the United States have tested positive for coronavirus.
    - They should find the proportion of people who have tested positive (about 0.00394), and then multiply that by the U.S. population (about 394 per 100,000).
- Ask students in their groups to discuss what this figure means, then report out.
    - If you sample people randomly, about 394 out of every 100,000 people would be expected to test positive for coronavirus.
    - This does *not* mean that 394 *must* test positively. The probability that a randomly selected person would test positive is about 0.394%.
- Ask students to imagine that the fraction of people with COVID-19 in their state is the same as the whole nation. They want to make a test that will help determine whether someone has coronavirus.
- All health tests have *false positive* and *false negative* rates. Ask the groups to discuss what these terms might mean.
    - A false positive is when someone does *not* have the disease, but the test says they do. A false negative result is the opposite.
    - False positive and negative rates are the probabilities of each result.
- Older adults, a vulnerable population, have a 10% probability of needing to go to the hospital if they get coronavirus.
- A laboratory has created a new test for coronavirus with a 10% false positive rate. Ask groups to find the probability that someone testing positive *actually has* the disease. (90 out of 100 or 90%)
- The lab wants to know the conditional probability that an older adult actually has the coronavirus and will need hospital care, *if they test positive*.
    - Encourage students to use the problem-solving strategy.
    - **Understand the problem.** Ask students to reiterate the problem. (What is the conditional probability that an older adult has the disease and will need go to the hospital if they test positive?)
    - **Develop a plan.** Ask the students if this problem is like any they have solved before. (Yes, it is like the diplomacy problem in SPT App. They should calculate conditional probability through multiplication.)
    - **Carry out the plan.** Ask students to discuss how they should set up the problem. (The probability an older adult with the disease will need to

go to the hospital is 10%. There is a 90% probability of having the disease if someone tests positive. These are multiplied to find a conditional probability of 9%.)

  - **Check the result.** Ask students to if the result seems plausible and why. (Yes: the conditional probability is lower than the original 10% because of false positives.)

- Ask each group to imagine their state lab has made a new test for coronavirus. Each group should find the conditional probability that a randomly selected older person who tests positive on the test would need to go to the hospital.

  - Randomly assign false positive rates of 8% - 35% to each group.

  - For example, if the false positive rate is 15%, the probability of an older person who tests positive who would need to go to the hospital is 10% * 85% = 8.5%.

  - Ask groups to share their findings and how the false positive rate impacts the conditional probability.

Classroom Activity 5.6: Retesting

To finish, students will communicate conditional probabilities in words.

- The state adopts a test that has a 5% false positive rate and a 2% false negative rate. For a public information campaign, the groups need to prepare for a press conference with information about the new tests.

- Groups should calculate the following figures to share at the press conference:

  - What does the false positive rate mean? If 10,000 people *without* the disease are tested, how many positive results are expected? (500)

  - What does the false negative rate mean? How many negative results are expected if 10,000 people *with* the disease are tested? (200)

  - Older adults have a 10% probability of needing hospital care if they have the disease. If an older adult tests positive, what is the conditional probability that they actually have the disease and will need hospital care? (9.5%).

  - Why is it important to keep working to develop tests with lower false positive and false negative rates?

- Stage a brief press conference. Ask each group to nominate one person to answer one of the questions in front of the class.

Possible alternative: This activity deals with coronavirus, a weighty issue. If this topic is problematic, an alternative is offered here.

The video game company from SPT App tests its games as they leave the factory to make sure they work. If a game is defective, there is a 10% chance it will not work in a player's game machine. These tests have false-positive and false-negative rates. Use the same figures and problem-solving techniques as shown in the activity. Reframe the importance of having precise tests as being beneficial to consumers.

Conclusion & Next Steps

Probabilistic thinking helps people make sense of the world around them. By completing **Supporting Probabilistic Thinking**, students now have a better understanding of probability, mathematical problem solving, and their real-world uses.

In middle school and high school, students will be exposed to statistics, the data-driven discipline that models our world. By completing this program, students will be able to appreciate the importance of probability in statistics, and they will be ready to use robust problem-solving techniques to tackle complex problems.

To extend students' learning from this program, and to prepare students for statistical learning, teachers may wish to consider incorporating some of these topics into classroom activities:

- *Mathematical problem solving.* The problem-solving technique presented in this program is robustness and broadly generalizable to many kinds of problems. By continuing to use this problem-solving process—or by adapting the technique to suit different problems—students will attack problems methodically.
- *Sampling techniques.* In this program, terms like "randomly selected" were used to indicate how observations were made in a random experiment. However, students can extend their knowledge by considering how sampling is performed in the real world (e.g., using random numbers to sample from a population, robocalling cell phones). As these sampling techniques deviate from the purely random, more caution must be used when interpreting data.
- *Basic descriptive statistics.* Students may already have some experience with descriptives such as averages, medians, and modes. With an understanding of probability, students will be better equipped to understand how these *measures of central tendency* all seek to describe a distribution of data. They are based in probability, and conversations about these descriptives and their use can help students in later studies.
- *Explicit references to probabilistic thinking.* Probabilistic thinking takes practice, just like any other skill. By asking students explicitly to use probabilistic thinking when considering probability, risk, or uncertainty, they will remember what they have learned in this program.

In parting, thank you for using **Supporting Probabilistic Thinking** in your classroom! Please visit the website for updates, news, and to tell us how the program has worked with your students.

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Appendix A: Student App Description

The SPT App is the tablet-based, student-facing application for **Supporting Probabilistic Thinking**. This section describes the features and functions of the SPT App.

General description of the SPT App

Note: At the time of this writing, the SPT App is still being planned. This appendix details the potential functions of the app.

The SPT App allows students to explore probabilistic thinking at their own pace. Students can use the app with minimal supervision: the activities help students learn independently. The app has several key features, summarized here.

- The activities reflect the **Supporting Probabilistic Thinking** framework with several engaging activities per topic.
- The app reinforces an explicit connection between probability topics and mathematical problem solving, the cornerstone of probabilistic thinking.
- App activities should be completed before the unit's topics are presented in the classroom. Afterwards, the app can reinforce learned knowledge and skills.
- A system of *quests* and *trophies* in the app provide positive reinforcement and help students celebrate their growth.

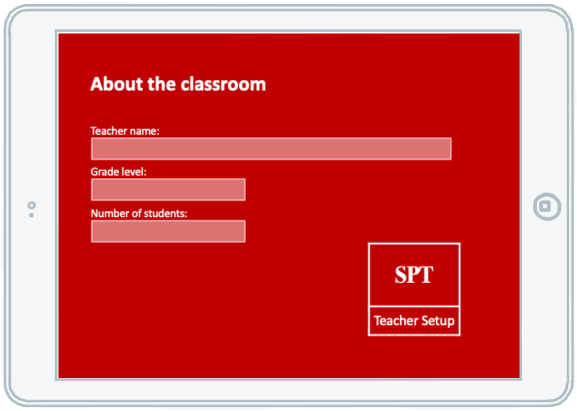


Figure 3. The teacher setup screen

Set-up and logging in

Installing. To set up the SPT App, download the app onto students' tablets. Teachers in schools with centralized control over tablets should contact their IT resource for advice.

Multiple students can share a single tablet and app, if needed. However, each tablet should be used by a single class or teacher.

Teacher setup. At the login screen, choose *Settings,* then *Teacher Setup.* The app will collect information about the classroom. The teacher must establish a 6-digit PIN to later access the *Teacher Dashboard.* To finish, choose Save on the *Teacher Setup* screen. The SPT App will then be ready to share with students.

Student setup. New students can begin using the app from the home screen. The home screen can be accessed at any time by choosing the house icon.

Figure 4. The student setup screen

New students should then choose New Student. On the *Student Setup* screen, students enter their name (or nickname) and choose an avatar. To finish, students choose Save and return to the home screen.

Logging in as a student. Returning students should choose their avatar from the home screen. New students see the introduction; all others are presented the *Quest Screen.*

Logging on as a teacher. To login as a teacher, choose Teacher Login from the home screen. The *Teacher Dashboard* screen is described later in this section.

Quests and trophies

Students navigate the app using the *Quest Screen*, shown in Figure 5. When students first use the app, only the first activity is unlocked. When a student engages with an activity—even if he or she cannot complete it completely—the next activity is unlocked.

As students engage with and complete activities, the activities are marked with one of three colors, as shown here.

- **Bronze**: The student has attempted the activity but has not yet completed it.
- **Silver**: The student has completed the activity but has room for improvement.
- **Gold**: The student has completed the activity successfully and in its entirety.

Students also earn trophies for completing (a) all of the activities associated with a given unit at the silver level or higher; (b) all of the activities associated with a given unit at the gold level; and (c) completing a fixed number of activities at the bronze level, at the silver level, and at the gold level.

Students receive their first trophy after engaging with the first activity (Activity 1.1: Introduction, described on page 24). To see their trophies, students can choose the trophy symbol from the home screen.

Teacher dashboard

The *Teacher Dashboard* shows students' progress in the SPT App. To access it, teachers choose *Teacher Dashboard* from the *Settings* menu and type the 6-digit PIN. For each student, the dashboard lists:

- Their process through the quest
- The time spent on activities in each unit
- The responses given to the reflection activity (described on page 26).

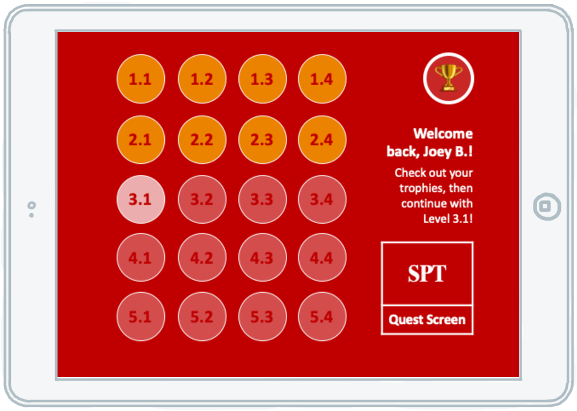
Students' responses to the reflection activity reveal their levels of understanding of probabilistic thinking (see page 5). Reviewing these responses allows teachers to assessing students' learning journeys.

Appendix B: Student App Activities

Activities form the heart of the SPT App. This section includes descriptions of each activity, their learning goals, and potential questions that students may have from each activity.

Four activities are provided for each unit, all accessible using the *Quest Screen.* Each unit ends with *Reflection*, described on page 26.

Figure 5. The quest screen



Activity 1.1: Introduction

When students use the SPT App for the first time, the *Quest Screen* prompts them to choose this introductory activity.

Learning goals. Students will (a) learn how to navigate the SPT App and earn trophies; and (b) learn about the importance of probabilistic thinking and randomness.

Scenario. In a nearby town, a video game company makes games that entertain millions. Today, the student is reporting for their first day of work as an intern.

Protocol. The president of the company welcomes the new intern. She explains that the student will help prepare games before they ship to players. She says the company has a way to track the student's progress.

The president shows the *Quest Screen* and describes how the student can choose activities. As the student completes activities, the student earns trophies which are shown by choosing the trophy symbol.

Back in her office, the president notes that most video games are based on probability, a way of describing how likely things are to happen. Probability is critical to games, so the student must use *probabilistic thinking* when testing games. Probabilistic thinking uses logic and problem-solving to think about probability in everyday situations.

The app briefly illustrates other scenarios in which probability and probabilistic thinking are used in real life, such as choosing the line at the supermarket which will be fastest, or when deciding whether to grab an umbrella when it looks like it might rain.

On this first day, the student will think about randomness, an important quality of many uncertain events. Random events are unpredictable in the short-term, but they form predictable patterns in the long-term.

At conclusion of the activity, the student is awarded a gold trophy. The app shows it under the trophy symbol, and then the student is left at the *Quest Screen*.

Activity 1.2: Shipments

Students begin exploring randomness by studying a classic example, the coin flip.

Learning goals. Students will (a) learn about two facets of randomness, short-term unpredictability and predictable long-term patterns; and (b) consider "fairness."

Scenario. While touring the factory, the student learns that copies of a new game are in short supply. After a crate of games is made, the foreperson on the factory floor flips a coin to decide whether the games should go to the Store A or Store B. The foreperson says this is fair because coin lands on heads and tails each half the time.

Protocol. The student is asked to imagine they are in charge of Store A or Store B. They flip a virtual coin several times, and animations of crates are shown going to stores. A tally sheet shows how many crates have gone to each store.

The student is asked to predict whether the next shipment will go to their store, to the rival store, or if it cannot be predicted; and why. (It cannot be predicted because each flip of the coin has an equal chance of landing heads or tails.) If incorrect, the student is given hints and support.

The foreperson offers to flip the coin 100 more times. An animation quickly flips the coin 100 times and the tally sheet updates.

The student is again asked to predict the destination of the next shipment and why. (It cannot be predicted, because each flip of the coin has an equal chance of landing heads or tails.) If incorrect, the student is given hints and support.

The student is asked to think about the next day's shipments and to predict about how many will go to each store. (About half are expected at each.) The tally is wiped clean, and 100 deliveries are simulated. The student checks his or her prediction.

The student is asked to think back to the foreperson's claim that flipping a coin is fair because it lands on heads and tails each about half the time. The student is asked to think about how one might check to see if that claim is correct. After a few seconds, the student is offered a suggestion: looking at the outcome of a few days' worth of flips.

The foreperson appears to simulate several days' worth of coin-flips. Tallies show the number of shipments made to each store. The student is asked to describe whether the data are consistent with the foreperson's claim about fairness. Then the last shipment ships and the activity ends.

Activity 1.3: Laundry

Students continue exploring randomness and consider the gambler's fallacy.

Learning goals. Students will (a) reinforce their intuitive sense of randomness; and (b) learn that prior observations cannot be used to predict events in a random process.

Scenario. While finishing the factory tour, the student is asked to retrieve uniforms from the laundry. Factory workers wear red, white, or blue uniforms, depending on their shift. The student gets a text asking for 10 more red uniforms for day shift workers.

Protocol. The student sees dozens of red, white, and blue uniforms tumbling in an industrial dryer. The student is reminded that he or she needs to collect 10 red uniforms. But because of the way the dryer is configured, the student can only retrieve one uniform at a time, the dryer door blocks the student's view as he or she reaches into the dryer to get a uniform, and the dryer tumbles the remaining uniforms in the dryer between picks.

The student chooses three uniforms. As the student puts them in color-coded baskets—one of each color, one at a time—a tally sheet shows the number of uniforms collected of each color, plus the colors of the three most recently selected uniforms.

Another intern emerges to help. He asks the student to predict the color of the next uniform; the intern predicts a different color than the student. The process repeats twice. The other intern remarks it is hard to predict the color of the next uniform because they are selected randomly.

After the student retrieves the next uniform, the app offers to flash forward eight picks: an animation shows the student quickly choosing uniforms, and the tally updates to show roughly equal numbers of uniforms in each color-coded basket.

The student is then asked to pick three more uniforms: all blue. The student receives another text asking for the 10 red uniforms. The other intern says he hopes the next uniforms are red. The student is then asked to predict the color of the next uniform: *blue*, because there's been a streak of blue; *red*, because there haven't been any reds in a while; or *impossible to predict*. After the next pick—a white uniform—the other intern reminds the student that you can't predict the outcome of a single event in a random process. The sequence in this paragraph repeats once.

The app offers to flash forward once more: uniforms are selected until each basket has nine uniforms. The student is asked to look back at the long-term pattern of selections, and to consider the proportions of uniforms of each color that are in the dryer. If the student hesitates, the other intern suggests that the long-term pattern might provide a clue. The app uses an x-ray animation to show the uniforms in the dryer, with equal numbers of uniforms in each color.

The student is asked to pick one more uniform. It is red, and the activity ends.

Activity 1.4: Reflection

Each unit in the SPT App ends with a reflection activity, and each has the same structure. This activity is detailed here.

Learning goals. Students will (a) revisit the probability topics and problem-solving techniques they have learned during the unit; and (b) engage with the information from a different perspective.

Philosophy. Probability is a new and highly abstract topic for most students. To learn about probabilistic thinking, one must practice it. A good way to reveal one's thinking process is to write thoughtfully on the topic. This activity asks students to write about what they have learned.

Protocol. Characters from the unit illustrate the main topics covered. For example, the factory foreperson reminds students about how coin-flips were unpredictable in the short-term but form predictable patterns in the long-term. Students are asked to think about what they knew before the unit and how much they have already learned.

The student is then presented the reflection prompts, one at a time. Students are told their teacher may review their responses.

Prompts. There are three prompts in each reflection activity. The first asks the student to apply a concept learned during the unit. The second asks the student to extend what he or she has learned. The third asks the student to reflect on his or her learning.

- The student flips a coin again and again. So far, the results are H, T, H, H, T, H, H, H. Alfaro thinks the next flip will definitely be tails because there have been fewer of them. Berta says it's impossible to know what the next flip will be. Who is right, and why?
- Charlotte is playing a new video game where the enemies can come out of Door 1, 2, or 3. She doesn't know if the movement of the enemies is random or not. She plans to play the game many times. What should she do to figure out whether the enemies' movement is random or not?
- What is one thing you learned in this unit that surprised you?

To help students respond to the prompt thoughtfully, the app asks students to write at least 20 words. If a student does not use a keyword from the unit (e.g., "random"), the app suggests that keywords may help.

By answering the first prompt, the next unit is unlocked on the *Quest Screen*. Students earn a trophy by answering all 3 prompts.

Teacher review. After students complete the reflection activity, their responses are shown on the *Teacher Dashboard.*

When reviewing students' responses, the probabilistic thinking framework (on page 5) can be used to help determine each student's level of understanding of the six key skills. These understandings will grow as students engage with the program.

Activity 2.1: Impossible

To begin their exploration of the sample space, students consider which events are possible in a series of simple experiments.

Learning goals. Students will (a) learn a simple strategy of how to determine if an event is possible (if the sample space is known); and (b) begin to use their imagination to envision a sample space.

Scenario. The lead game designer is showing the student scenes from several upcoming games. He notes that game designers have to think through all the potential outcomes of any game action, and whether an event is possible or impossible.

Protocol. The designer says that the proper term of all the possible events of a process or experiment is called the sample space. An event is possible if it is in the sample space and is impossible otherwise.

Ten experiments are illustrated, and the designer asks the student whether an event is possible or impossible. Examples include:

- The flip of a fair coin. The sample space is heads and tails. Is it possible for the outcome to be “heads”? (Yes.)

- The roll of a single die. The sample space is 1, 2, 3, 4, 5, or 6. Is it possible for the outcome to be 7? (No.)
- A shot-putter always throws the shotput 7–18 yards from the base. Is it possible for him to throw 11 yards? (Yes.)

After each experiment is illustrated, the student is asked to imagine the sample space. It is then listed on screen.

Activity 2.2: Listings

Students are shown several experiments and list the sample space systematically.

Learning goals. Students will (a) learn a technique for enumerating the sample space; and (b) learn to check whether they have listed all the possible events.

Scenario. Back in his office, the lead game designer reiterates that knowing the sample space is important to designing a game. He challenges the student to a contest using the sample space of different experiments.

Protocol. The designer tells the student that one way to figure out the sample space is to list out the possible outcomes from lowest to highest (or from first to last). He races the student in listing out the sample space for six experiments. Examples include:

- The roll of a single die. What is the sample space? (1, 2, 3, 4, 5, and 6.)
- A vowel is picked at random from a bag of Scrabble tiles. What is the sample space? (A, E, I, O, U. Y is optional.)
- Two dice are rolled. What is the sample space of sums? (Numbers 2–12.)

After each experiment is described, the student is asked type in the possible outcomes, one at a time. Each time the student types in an event in the sample space, a car with the student’s name advances. (The designer’s car advances once every 8 seconds.) The race ends when the designer wins or when the student lists all possible the events.

If students hesitate, they are asked if they have listed all of the events. The designer suggests asking whether there are any smaller or larger values that are possible.

Activity 2.3: Frankenstein

Students are given additional practice in thinking about the sample space, and they learn a technique to map out the sample space for multi-stage experiments.

Learning goals. Students will (a) learn a technique that can be extended to find the sample space for multi-stage experiments; and (b) use multiplication to find the total outcomes of a multi-stage experiment.

Scenario. The lead game designer asks the intern to use a template to build monsters to use as opponents in an upcoming game. He asks the intern to create as many opponents as possible using the template.

Protocol. An artist has created a “template” for monsters. In the first scenario, monsters can have one of three heads and one of four bodies. The app shows the possibilities for heads and bodies.

The student is asked to use these heads and bodies to create as many unique monsters as possible. A grid appears with three rows and four columns. The student is asked to choose all of the unique head-body pairs.

The student chooses head-body pairs, one at a time. If the student has not yet selected the pair, the head and body combine, animate, and walk to the grid. Monsters with a given head are all shown in the same row, and monsters with a given body are all shown in the same column. If a student has already selected a head-body pair, the student is prompted to select again.

After the student has filled the grid, the student is asked how many monsters were created (3x4=12, as shown on screen).

A new scenario appears: a sea monster must be built with one of five heads and two tails. The student is asked to predict the number of monsters that can be built from the template. The student completes the grid and checks his or her answer.

A third and final scenario appears: a robot must be built with one of three heads, two bodies, and three pairs of feet. The protocol is the same as the first scenario: the student completes a grid, and the student is asked how many robots were created (3x2x3=18).

Activity 2.4: Reflection

Each unit in SPT App ends with a reflection, and each has the same structure. See the description of Activity 1.4 for a synopsis.

Prompts. There are three prompts.

- Dante reaches into a bag of colored chocolates without looking and pulls out a yellow chocolate. Because the bag has yellow and brown candies, Ernesto says that the next one she pulls out of the bag will be brown because the colors will probably alternate. Is Gloria correct? Why or why not?
- Gloria is very hungry, so she is going to make a sandwich randomly. She reaches into the fridge to get one of three meats without looking: turkey, beef, and tofu. She also grabs one of three vegetables without looking: lettuce, tomato, and onion. How many different sandwiches can she make if her sandwich will have one meat and one vegetable? How many can she make if her sandwich has one meat and *two* vegetables?
- What is one thing you learned in this unit that surprised you?

Activity 3.1: Unlikeliest

Even without instruction, students likely have their own intuitions about events which are likely or unlikely to occur. This activity encourages students to explore intuitions. These understandings provide a scaffold to quantifying probabilities, presented later in this unit.

Learning goals. Students will (a) bolster their intuitive understandings of which events are likely and unlikely in several experiments; and (b) use these intuitions to make simple decisions.

Scenario. A game artist asks the intern to help with a new adventure game. The game will ask the player to make choices along the way, all selected randomly by the game.

She asks the intern to prepare by playing some of the prototype mini-games, and she encourages the student to score high.

Protocol. One at a time, the student is presented with a mini-game (from the adventure game) based on probability. In each, the student must select events which are likely (or unlikely) to occur. The first mini-game is shown here.

- A spinner with four colors: red takes 45% of the available space, blue 30%, green 20%, and yellow 5%.

Each portion of the spinner is associated with a different part of the adventure game (e.g., blue for a water level). The student activates the spinner twice to see it work.

The student is asked to predict the color that the spinner will land on the most. If the student hesitates, the student is asked to consider which color takes up the most space. After the prediction is made, the spinner activates 20 times: a tally sheet records the colors selected. If the student was correct, the game continues; otherwise the colors scramble and the game repeats.

The student is then asked to select the color that the spinner will likely land on the least. The spinner activates and tallies as before. The game continues with several additional mini games described here.

- A helicopter flies over several rooftops and drops a payload which can be carried unpredictably by the wind. Building A takes up 4 city blocks, Building B takes 2 blocks, Building C takes 1.5 blocks, and Building D takes 0.5 blocks.
- A fisherman uses several nets, and fish can be caught in any net. Net 1 is 85 m^2, Net 2 is 60 m^2, Net 3 is 60 m^2, Net 4 is 35 m^2, Net 5 is 25 m^2, and Net 6 is 10 m^2.

To make their decisions, the student is reminded to ask themselves which events are likeliest by considering which possibilities take the most space, the most area, or are most frequent.

Activity 3.2: Caves

Quantifying probabilities is often important when making decisions. This activity helps students probabilities quickly.

Learning goals. Students will (a) calculate probabilities for single-stage experiments; and (b) express these probabilities using numerical expressions.

Scenario. The artist asks the student to try a new level in the adventure game, based on escaping from caves in an ancient tomb.

Protocol. A map shows the student in a booby-trapped system of six caves. To escape from a cave, the student must enter the combinations before time expires. A cave's combination is the probability of a given event occurring, given as a fraction. Each cave requires two combinations.

The scenario for the first cave is shown here. Subsequent caves are similar.

- Cave paintings show five archers, three runners, and one swimmer.

The student is told that an arrow will be launched from the far wall, and it will randomly strike one of the cave paintings.

To escape, the student must enter the probability for two possibilities, expressed in the form "___ out of ___." The symbols and correct responses are shown here.

- Probability for an archer (5 out of 9)
- Probability for a swimmer (1 out of 9)

In the first cave, the artist (in voiceover) gives the student advice on how to obtain each probability. She notes that the first number is associated with the number of target events (e.g., archers), and the second number is associated with the total number of possible events (e.g., archers + runners + swimmers). She helps the student fill in the probability for the archer, and then asks the student to fill in the probability for the swimmer. Afterwards, she tells the student to escape from the remaining caves using the same technique.

Activity 3.3: Safari

Quantifying probabilities takes practice. This activity provides additional practice in calculating probabilities, and students will express probability in different ways.

Learning goals. Students will (a) practice calculating probabilities for single-stage experiments; and (b) express probabilities using multiple representations.

Scenario. The game artist says that the "photo safari" level is the hardest in the new game, and they would like to find out why. She asks the student to examine the probabilities associated with the game.

Protocol. The student is shown a gentle savannah with grass and trees. The artist (in voiceover) explains that the goal is to take a picture of animals that roam by. Because the way the camera controls work, there is an unpredictable amount of time between pushing the shutter and taking a picture: when the shutter is pushed, it is not possible to predict the position of any animal when the picture is actually taken.

The artist explains the scoring system. The screen is divided into 12 zones of equal area (in a 3x4 grid). The player can choose any one of the 12 zones to place the camera.

Groups of animals are shown, one group at a time. Each animal takes up one zone on screen: each zone contains either zero or one animal at the time the picture is taken. The animals and number of animals which appear per group are shown here.

- Giraffe (4 per group)
- Parrot (2 per group)
- Hippopotamus (6 per group)
- Gazelle (1 per group)

The student is given two tries to take a picture of each animal group. The student is then asked to report the probability of capturing an animal's picture in "___ out of ___" form. If the student hesitates, the artist (in voiceover) suggests that the student count the number of animals which appear and the total number of zones (e.g., four giraffes appear on screen with one giraffe per zone; there is a 4 out of 12 chance of taking the picture of a giraffe).

After the student expresses the probability in "___ out of ___" form, the value is transformed into a stacked fraction (e.g., "4 out of 12" becomes "$^{4}/_{12}$"). The artist tells the student that probability is often shown this way, as a fraction. The artist then asks the student if there is a way to simplify the fraction (e.g., from "$^4/_{12}$" to "$^1/_3$").

The activity continues until the student describes the probabilities (in both forms) for all four groups of animals.

Activity 3.4: Reflection

Each unit in SPT App ends with a reflection, and each has the same structure. See the description of Activity 1.4 for a synopsis.

Prompts. There are three prompts.

- Heidi is playing a word game where she chooses letter tiles from a bag. The bag has one tile for every letter of the alphabet. Without looking, she takes out one tile. What is the probability that her tile is a vowel, and why?

- Isaac wants a cookie from the cookie jar. Inside the jar, there are 6 chocolate chip cookies and 4 oatmeal cookies. The cookies are mixed up, and Isaac gets one cookie at a time without looking. What is the probability that he gets a chocolate chip cookie? If he gets a chocolate chip cookie on his first pull and eats it right away, what is the chance that he gets an oatmeal cookie on his second pull? Why?

- What is one thing you learned in this unit that surprised you?

Activity 4.1: Recycling

The skill of comparing probabilities can be scaffolded by presenting problems visually. This activity helps students compare the probabilities of events and make decisions.

Learning goals. Students will (a) compare probabilities associated with events in a single-stage experiment; and (b) make decisions based on these comparisons.

Scenario. In the factory, the foreperson asks the student to recycle some old game posters make room for new ones. To make it a bit more exciting, the foreperson asks the student to drop the old posters into the recycling bins with a special trash launcher.

Protocol. Pairs of recycling bins are visible from a third-floor window. The foreperson (in voiceover) explains that the game designers created the trash launcher to encourage recycling within the company. Once loaded with recyclables, the student must point the launcher at the recycling bins and hit the "launch" button. The launcher hurls the recycling into the air, and the posters flutter toward the bins below.

A top down view of the bin bay appears. In a 3x4 grid, the screen is divided into 12 zones of equal area. When launched, posters have an equal chance of landing in each zone.

For five tasks, two bins are presented:

- 3x2 rectangle vs. 3x1 rectangle
- 2x2 square vs. 1x1 square
- 3x1 rectangle vs. 5-unit L-shape
- 2x2 square vs. 3x1 rectangle
- 5-unit L-shape vs. 2x2 square

Three questions are posed for each task:

- What is the probability that the recycling will land in the *first* bin? The *second* bin?
- What is the probability that the recycling will land in *neither* bin?
- When launched, will the recycling most

likely land in the first bin, the second bin, or neither bin?

Students are asked to report probabilities in "___ out of ___" form (e.g., 6 out of 12). If the student hesitates, the foreperson (in voiceover) suggests that the student count the number of zones occupied by a bin, and the number of total zones (e.g., a bin takes up five zones; there is a 5 out of 12 chance of trash randomly landing in it).

The student gets one point for every poster that lands in the chosen bin (or neither bin). The student launches the three posters by pressing a button, and the system launches another 21 posters for a total of 24. The system then counts posters that fell into the student's selected bin before continuing.

Activity 4.2: Fishing

Comparing probabilities is harder when experiments have different numbers of possible outcomes. In this activity, students practice comparing probabilities from experiments with different sample spaces.

Learning goals. Students will (a) compare probabilities associated with events in two single-stage experiments with differently sized sample spaces; and (b) avoid common pitfalls when making such comparisons.

Scenario. The foreperson gives the student a well-earned break. Another intern invites the student to go fishing behind the factory. They grab fishing rods and head off.

Protocol. Small fishing lakes dot the field behind the factory. The other intern (in voiceover) says that a fish, the red snipsnap, inspired the firm's first hit, *Fish Crossing.* Together, they will catch (and release) all the red snipsnaps they can.

Two lakes are shown from above, each taking up half of the screen. Fish of different colors visibly swim in both ponds, including red, yellow, and blue snipsnaps.

The student is told that the pair can only fish from one lake at a time; and with their simple fishing rods, they have an equal chance of catching any fish in the lake. After catching each fish, they will take a selfie with it before releasing it and fishing again. They want to catch red snipsnaps, so they must choose the lake that with the best chance of catching that type of fish.

For eight trials, two lakes are shown at a time. Examples include:

- *Lake 1:* 5 red and 5 yellow snipsnaps
  Lake 2: 5 red and 10 yellow snipsnaps
- *Lake 1:* 4 red and 4 blue snipsnaps
  Lake 2: 8 red and 16 blue snipsnaps
- *Lake 1:* 3 red and 5 yellow snipsnaps
  Lake 2: 6 red and 16 blue snipsnaps

Three questions are posed for each trial:

- What is the probability of catching…
  - …a red snipsnap in Lake 1?
  - …a red snipsnap in Lake 2?
- Which lake has a higher probability of catching a red snipsnap?

Students are asked to report probabilities in "___ out of ___" form (e.g., 5 out of 10). If the student hesitates, the other intern (in voiceover) suggests that the student count the number of red snipsnaps and the total number of fish (e.g., a lake has 5 red fish and 10 total fish; the probability is 5 out of

10). Probabilities are shown as fractions (e.g., $^5/_{10}$) that overlay each lake.

If the student hesitates when choosing the lake with the higher probability, the other intern tells them (in voiceover) to compare the probabilities and choose the larger one. When applicable, the other intern suggests (a) *not* just choosing the lake with the most red fish, but instead choosing the lake with the highest *proportion* of red fish; and (b) simplifying the fractions using common denominators, then comparing numerators.

After choosing a lake, the student is shown casting a rod and catching a fish. If they chose the lake with the higher probability, they catch a red snipsnap; otherwise, a different color. A selfie is shown with their fish. After each trial, they are reminded that the best lake to choose is the one with the highest *proportion* of red fish, not always the one with the highest *number* of red fish.

Activity 4.3: Malfunction

Comparing probabilities is essential when figuring out whether an experiment is fair. In this activity, students collect data and compare probabilities to determine fairness.

Learning goals. Students will (a) observe and summarize data from an experiment; and (b) compare probabilities to determine whether an experiment is fair.

Scenario. The foreperson welcomes back the intern from fishing. She says some of the printers are acting up; and because she knows the student is good with numbers, she would like the student to look at them.

Protocol. In the print room, seven conveyor belts are shown. Each carries a combination of 10 game boxes, each of which is blue for *Fish Crossing* or red for *Fish Crossing 2.* Printers churn out boxes which travel along the belt to the other side of the room.

The foreperson (in voiceover) says that each time a printer makes a box, it randomly chooses whether to create a red or a blue box. The probability for each color should be *1 out of 2:* that is, each color should have an equal chance of being printed. However, printers are faulty, and the probability may be different for some. The student needs to find which printers are malfunctioning.

The student will examine four printers. The probability of printing a red box for each is:

- Printer 1: 1/2 probability (normal)
- Printer 2: 1/4 probability (malfunction)
- Printer 3: 1/2 probability (normal)
- Printer 4: 3/5 probability (malfunction)

These probabilities are kept secret from the student. One printer and belt at a time, the student sees the experiment in action: the printer creates red or blue boxes according to its (secret) probability, and the student sees 10 boxes pass by on the conveyor belt.

The foreperson (in voiceover) asks the student how many boxes out of 10 would be expected to be red if the printer were functioning properly (5), then how many blue (5). The student is reminded that a printer *should* have a 1 out of 2 chance of printing a red box each time. If the student hesitates, the foreperson suggests changing “1 out of 2” into the form “____ out of 10.”

The student is then prompted to count the red boxes on the belt. The system shows if the count is higher, lower, or the same as the student's prediction (e.g., "out of 10, one more than expected"). This process repeats four times (for a total of 50 boxes) and the sum are shown (e.g., "48 out of 50, 2 fewer than expected").

The student is then asked to determine whether the printer is malfunctioning. The foreperson (in voiceover) suggests they look at the number of red boxes observed, and to use it to estimate the probability the printer is using. If the student hesitates, the foreperson reminds the student that a random process tends to be predictable in the long-term: if the printer works properly, the long-term count may not be exactly 25 out of 50, but it will likely be close.

After the student determines whether the printer is malfunctioning, the printer's secret probability is revealed. If deemed malfunctioning, the student pulls a lever to stop the press. The activity then repeats.

Activity 4.4: Reflection

Each unit in SPT App ends with a reflection, and each has the same structure. See the description of Activity 1.4 for a synopsis.

Prompts. There are three prompts.

- Launa reads online that there is a 6 out of 10 chance of rain tomorrow. Later, on TV, she hears that there is a 60% chance of rain. Are these two probabilities the same or different, and why? If you were Launa, would you expect rain?
- Jamie and Kyle are playing a number game. They write the numbers 1 through 10 on scraps of paper and put them in a bowl. They will pick a number randomly. Jamie will win if the number is 6 or less. Kyle will win if the number is even. Who has the best chance of winning and why? Can both win? Can both lose?
- What is one thing you learned in this unit that surprised you?

Activity 5.1: Spaceship

Before considering conditional probability, one must first consider correlation. In this activity, students will practice observing and describing correlated events.

Learning goals. Students will (a) make observations to identify correlation; and (b) describe the correlation between events.

Scenario. In the marketing department, the director says that they are about to release a new space game. Before they do, they need to put some playing tips on the game website. The director asks the student to play the game and answer some questions.

Protocol. From the bridge of a spaceship, the student sees the vacuum of space with glimpses of spaceships on the horizon. The marketing director (in voiceover) explains that this game has three types of aliens:

- *Rocknoids* are destroyed by paper beams
- *Scissorsaurans* are destroyed by rock-ets
- *Papersons* are destroyed by scissor jabs

The student can fire the weapons using on-screen buttons. The goal is to destroy the aliens, but the student can only see what type of alien is coming when they are close. Any of the aliens can come in red, blue, green, or silver spaceships; but some aliens have preferences (described below).

In Wave 1 of alien attacks, the marketing director asks (in voiceover) what color spaceship Rocknoids tend to use. In the wave of 12 attacks, Rocknoid make six. Rocknoids always pilot red ships, and other aliens never pilot red ships. After the wave, the student is prompted to report the color.

In Wave 2, the director asks what color ship Scissorsaurans tend to use, and he warns they may use more than one. In the wave of 12 attacks, Scissorsaurans make six; for five, they pilot silver ships. After the wave, the student is prompted to report the color. The director says this is a *correlation* between Scissorsaurans and silver ships.

In Wave 3, the director asks which type of alien Papersons tend to follow. In this wave of 12 attacks, Papersons make four; three of these attacks are preceded by Rocknoids. After the wave, the student is prompted to report the type of alien.

In Wave 4, aliens attack in pairs. The director asks the student to describe the correlation between Rocknoids and Papersons: often together, rarely together, or purely random. In this wave of eight attacks, Rocknoids and Papersons are paired in six. After the wave, the student is prompted to describe the relationship.

In Wave 5, the director asks the student to describe the correlation between Papersons and silver spaceships: often together, rarely together, or purely random. Papersons make five of eight attacks in this wave; only one is in a silver spaceship. After the wave, the student is asked to describe the relationship.

Activity 5.2: Envoy

Using conditional probability relies on knowing how one event affects the likelihood of another. In this activity, students explore this concept.

Learning goals. Students will (a) describe how one event's probability can be affected by another's; and (b) understand that events can become impossible in non-replacement experiments.

Scenario. The marketing director tells the student that they are getting some great tips to put on the website. He asks the student to keep playing.

Protocol. Earth and the aliens inch toward peace. The student meets an alien envoy to make a peace offering of chocolate candy.

A box of 12 candies is shown in a 4x3 grid, each labeled. On the top row, all four candies are milk chocolate; white chocolate in the middle, and dark chocolate on the bottom. The candies in the first column are filled with caramel; followed by almond, cherry, and peanut in subsequent columns.

The director (in voiceover) says the envoy randomly chooses candies by beaming them into his or her mouth one at a time. The student needs to keep Earth command appraised so they can activate additional chocolate candy stockpiles, if needed.

The envoy picks three candies: caramel dark chocolate, almond milk chocolate, and milk chocolate caramel. Earth negotiators asks the student to report the chance that the next candy selected would be…

- milk chocolate (2 out of 9)

- almond filled (3 out of 9)
- white chocolate *and* cherry (1 out of 9)

If the student hesitates, the director (in voiceover) suggests the students count the number of candies remaining that meet the criteria and the total number remaining.

The envoy selects three more candies: cherry dark chocolate, cherry milk chocolate, and milk chocolate peanut. Earth negotiators ask the student to report the chance that the next candy would be…

- milk chocolate (0 out of 6)
- almond filled (3 out of 6)
- white chocolate *and* cherry (1 out of 6)

If the student hesitates, the director (in voiceover) suggests the students count the number of candies remaining that meet the criteria and the total number remaining.

Earth negotiators ask the student to explain the findings. She asks the student:

- What does it mean that there is a 0 out of 6 probability of selecting milk chocolate next? (It is impossible.)
    - Why is it impossible? (Because they have all been eaten.)

- Which is more likely: that the next candy will be almond filled, or the next candy will be cherry filled? (Almond.)
    - Why almond? (The probability is 3 out of 6; which is higher than the probability for cherry, 2 out of 6.)

Earth negotiators tell the student to expect a new box of candies: the candies appear and replace the old box. The level repeats with different selections by the envoy.

Activity 5.3: Diplomacy

By knowing the probability of one event, one can better estimate the probability of a related event. In this activity, students practice this skill quantitatively and make decisions based on conditional probability.

Learning goals. Students will (a) calculate conditional probabilities; and (b) use conditional probabilities to make decisions.

Scenario. The marketing director says that players will win a prize when they finish the game. But to prioritize the creation of the prizes, the department needs to know about the likely ways the game could end.

Protocol. At peace talks, the student will negotiate with one of 20 alien leaders. Earth may gain an advantage if the student can predict which of the alien leaders will be selected to negotiate. The alien bloc will choose qualities they want in a negotiator, and they will pick a negotiator randomly from those with those qualities. Earth intelligence is working to determine which qualities have been selected. The marketing director (in voiceover) tells the student that he or she should use that information to identify the likely negotiating partner.

Pictures of the alien leaders form a 5x4 grid. Four qualities are listed under each picture:

- *Galaxy:* Alpha, Beta, Gamma, Delta, Echo
- *Main concern:* Arms, trade, food, schools

The five alien leaders in a given row are all from the same galaxy. All the leaders in a column have the same main concern.

Earth intelligence asks for an initial report of probabilities, if the negotiator were selected purely randomly, that they would:

- be from Alpha galaxy
- be concerned with trade
- focus on trade *and* from the Alpha galaxy

If the student hesitates on the last question, the director (in voiceover) suggests that (a) the student should count the number of aliens with both qualities, and (b) one can also multiply the first two probabilities.

An intelligence report then flashes onto the screen that says that the bloc will choose a negotiator from Alpha or from Echo galaxy. The student is asked for these probabilities:

- The probability that the negotiator will be from the Alpha *or* Echo galaxy…
    - *and* mainly concerned with trade
    - *and* mainly concerned with arms

If the student hesitates, the pictures of the leaders from other galaxies dim, and the director (in voiceover) suggests that the student count the number of leaders who meet both criteria and the total number of leaders (e.g., trade-focused) to express the probability in "___ out of ___" form.

A new intelligence report says the bloc has identified 20 different potential leaders, but their backgrounds are less predictable. Their galaxies and main concerns are not evenly distributed as before (e.g., different numbers from each galaxy).

Intelligence reports that the negotiator will be from the Gamma galaxy. The student is asked for the probability that the negotiator will be from Gamma *and* will have each of the four main concerns (e.g., arms). If the student hesitates, the pictures of the leaders from other galaxies dim, and the director (in voiceover) suggests that the student count the number of Gamma leaders who have each main concern. The student is then asked to select which of the four main concerns would be the best for Earth negotiators to prepare to discuss.

The game ends with an image of a peace treaty student signed by the student and the negotiators. The director says that this image is the prize for winning the game.

Activity 5.4: Reflection

Each unit in SPT App ends with a reflection, and each has the same structure. See the description of Activity 1.4 for a synopsis.

Prompts. There are three prompts.

- Marcus and Nicole are playing a number game. Marcus picks a number between 1 and 100 and asks Nicole to guess it. He writes it down, and Nicole sees it has a "5" in it. Nicole randomly picks a number from all numbers between 1 and 100 with a "5" in them. What is the chance she guesses Marcus' number? Why?
- At the grocery store, there are eight kinds of breakfast cereal: four are made from rice, three from wheat, and one from quinoa. Five of the boxes are blue, two are yellow, and one is brown. Omar asks his sister to pick one box randomly from the shelf. Which is more likely: that she picks a wheat cereal in a yellow box, or that she picks a quinoa cereal in any box? Why?
- What is one thing you learned in this unit that surprised you?

Appendix C: Glossary

conditional probability. The probability that one event will occur when the probability of another event is known.

correlation. The tendency for two related events to occur together. Often quantified with a value of -1 to 1.

event. The outcome of an experiment. May be a single outcome (e.g., *heads* when a coin is flipped) or a set of outcomes (e.g., *heads-heads-tails* when a coin is flipped three times in a row).

experiment. A process that yields an event. In this context, an experiment is not necessarily a scientific experiment, but is simply a discrete act that yields events. Sometimes called a process or game.

fair. A game (or experiment) is fair if the long-term pattern of results does not favor one player over the other. For example, a coin-flip is a fair game for two people because each side will win about half the time.

gambler's fallacy. An error in logic where one believes the next event in a random sequence can be predicted from those already known.

observation. An observed event.

probabilistic thinking. The use of intuitive and quantitative ideas of chance, paired with logic and mathematical problem-solving techniques, to consider real-world questions of uncertainty and risk.

probability. The chance that an event will happen. Often expressed as a number between zero and one (e.g., 0.75, 75%). Also, a branch of mathematics associated with uncertainty.

random. A quality of events where a future event cannot be predicted from past events. Random events are unpredictable in the short term (i.e., for only a few observations), but tend to reveal predictable patterns over the long-term (i.e., over many observations).

sample space. The set of all possible events (outcomes) of an experiment.

statistics. The scientific discipline that uses probability to create models for real-life data. The study of probability is typically seen as a branch of mathematics, and statistics is a companion discipline.

wishful thinking. An error in logic where one believes one can influence a random process through hopes or wishes.